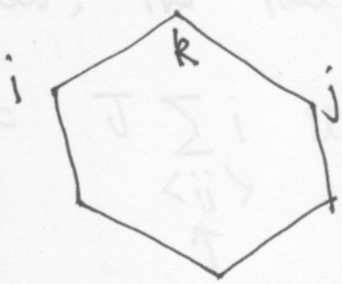
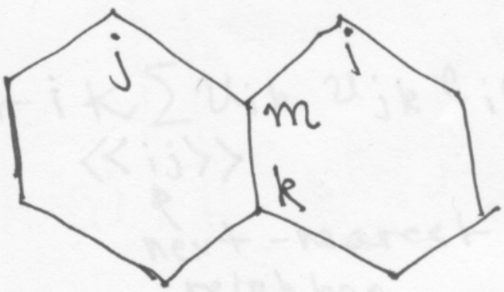
The Non-abelian B-phase

Consider perturbing the Hamiltonian $\sum XX + YY + ZZ$ with $V = -\sum [h_x X + h_y Y + h_z Z]$. This breaks time reversal. The effective Hamiltonian is

$$H_{eff} = -\frac{h_x h_y h_z}{J^2} \sum_{ijk} X_i Y_j Z_k$$

where i, j, k belong to the following two geometries:

(a) 

(b) 

\+ all symmetry related configs.

Let's consider (a) first:

$$X_i Y_j Z_k = i b^x_i c_i \, i b^y_j c_j \, i b^z_k c_k$$

$$= i b^x_i c_i \, i b^y_j c_j \, i b^y_k b^x_k$$

(recall $b^x b^y b^z c = 1$)

$$= (i b^x_i b^x_k)(-i b^y_j b^y_k) \, i c_i c_j$$

$$= -i u_{ik} u_{jk} c_i c_j$$

Next consider (b):

$$X_i Y_j Z_k = i b^x_i c_i \, i b^y_j c_j \, i b^z_k c_k$$

$$= i b^x_i c_i \, i b^y_j c_j \, i b^z_k c_k \underbrace{b^x_m b^y_m b^z_m c_m}_{(=1)}$$

$$\propto i u_{im} u_{jm} u_{km} c_i c_j c_k c_m$$

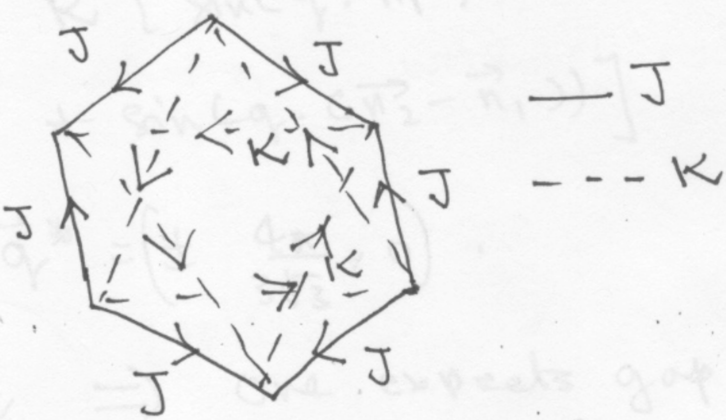
Assuming that the perturbation is small ($|h| \ll J_x, J_y, J_z$), one can neglect the (b) contribution since it corresponds to 4-fermion interaction.

Thus, the full effective Hamiltonian is

$$H = J \sum_{\langle ij \rangle} i \, u_{ij} c_i c_j + i K \sum_{\langle\langle ij \rangle\rangle} u_{ik} u_{jk} c_i c_j$$

nearest neighbor. next-nearest neighbor

$$K \sim \frac{h_x h_y h_z}{J^2}.$$



Since this is still a fermion coupled to a $\mathbb{Z}_2$ guage field, the ground state will again lie in the sector where $\prod_{\hexagon} U = 1$ ("vortex-free" sector) and hence one can set $U_{ij} = 1$.

Now the band structure becomes:

$$H = [c^{\dagger}_A(q) \quad c^{\dagger}_B(q)] \begin{bmatrix} \Delta(q) & if(q) \\ -if^*(q) & -\Delta(q) \end{bmatrix} \begin{bmatrix} c_A(q) \\ c_B(q) \end{bmatrix}$$

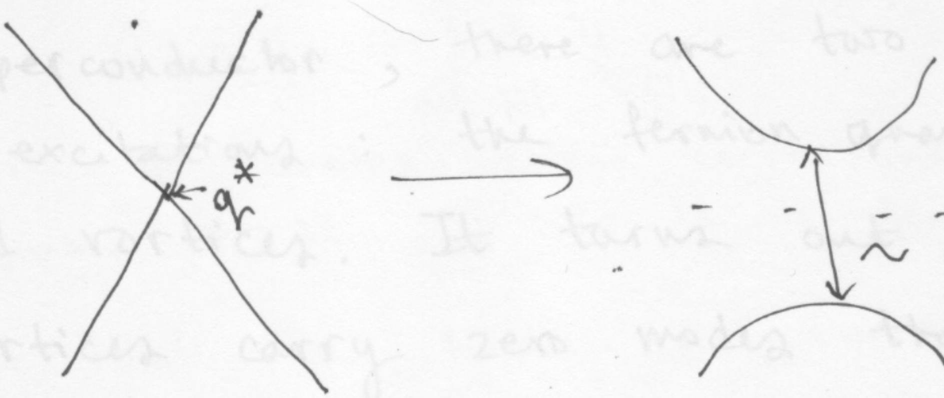
where $\Delta(q) \propto K [\sin(\vec{q}\cdot\vec{n}_1) - \sin(\vec{q}\cdot\vec{n}_2) + \sin(q\cdot(\vec{n}_2 - \vec{n}_1))]$

Close to $\vec{q} = \pm\vec{q}^* = \left(\pm \frac{4\pi}{3\sqrt{3}}, 0\right)$.

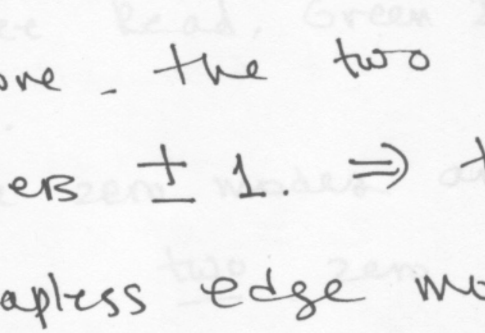
$\Delta(\vec{q}) \sim K \Rightarrow$ one expects gap opens up in the spectra.

Close to $\pm\vec{q}^*$, $\varepsilon(\vec{q}) \simeq \pm\sqrt{J^2|\delta q|^2 + \Delta^2}$

where $\delta q \sim q - q^*$ and $\Delta \propto K$.



Most interestingly, the band structure is identical to a $p_x + ip_y$ superconductor and therefore the two bands carry Chern numbers ± 1. $\Rightarrow$ the system supports gapless edge modes which are neutral and chiral with chiral central charge $c_- = \frac{1}{2}$



These modes can be detected by energy transport measurements e.g. K_{xy}.

Vortex zero modes / Unpaired Majorana modes :

Since the system resembles a $p_x + ip_y$ superconductor, there are two kinds of excitations : the fermion quasiparticles and vortices. It turns out that vortices carry zero modes that are neutral. Infact, the edge modes can be thought of as zero modes in a thick vortex. See Read, Green 2000 for a derivation.

Since these zero modes are neutral (Majorana), two zero modes ~~can~~ have the same Hilbert space as that of a complex fermion. ~~One can~~ Given two vortices, each carrying one Majorana zero mode (MZM), one can either fill this single complex fermion state or not.

These two possibilities are represented by the following <u>fusion</u> rule:

$$\sigma \times \sigma = \underset{\substack{\uparrow \\ \text{not fill}}}{1} + \underset{\substack{\nwarrow \\ \text{fill}}}{\psi}$$

where σ denotes anyon corresponding to a single MZM and ψ is a fermion.

The fact that there are two possibilities for fusion makes σ a non-abelian anyon. The Hilbert space dim for $2n$ σ's is 2^{n-1} if one restricts to a fixed fermion parity.

$\Rightarrow$ 'quantum dimension' $d_\sigma = \sqrt{2}$. The other fusion rules are

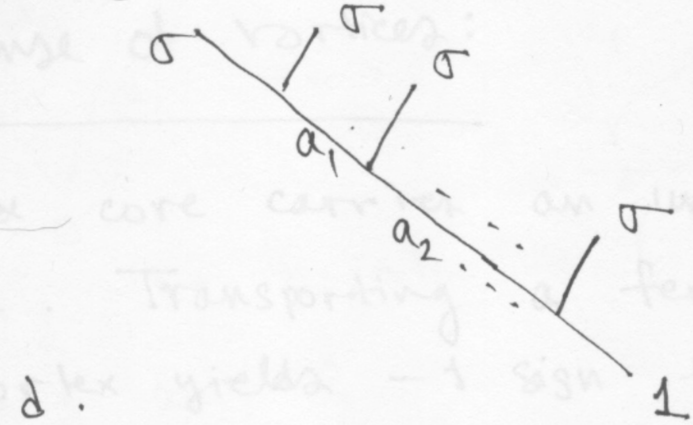
$$\psi \times \psi = 1$$

$$\sigma \times \psi = \psi \times \sigma = \sigma$$

In general, one writes

$$a \times b = N_{ab}^{c} \, c$$

where a, b, c are anyon types. Let's consider fusing several σ's repeatedly such that the final outcome is 1 (vacuum). This allows one to calculate ground state degeneracy in the presence of several σ anyons. One represents this by the diagram:



g.s.d.

$$= \sum_{a_1,\dots,a_{n-1}} N_{\sigma\sigma}^{a_1}\, N_{a_1\sigma}^{a_2} \cdots N_{a_{n-1}\sigma}^{1}$$

Define $N_{i\sigma}^{j} \equiv M_{ij}$

$$\Rightarrow \text{g.s.d.} = \cancel{[N^n]_{\sigma 1}} = [M^n]_{\sigma 1}$$

As $n \to \infty$, the g.s.d. will be dominated by the largest eigenvalue of M.

Explicitly,
$$M = \begin{array}{cc} \begin{array}{ccc} 1 & \sigma & \psi \end{array} & \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} & \begin{array}{c} 1 \\ \sigma \\ \psi \end{array} \end{array}$$

$$\operatorname{eig}(M) = \pm\sqrt{2},\ 0$$

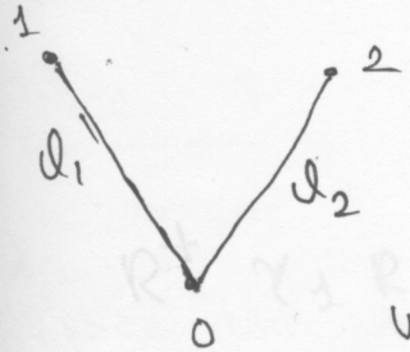
$$\Rightarrow \text{g.s.d.} \sim 2^{n/2} = d_\sigma^{\,n}$$

where $\boxed{d_\sigma = \sqrt{2}}$

Since ψ has only one fusion channel, it is abelian and has $d_\psi = 1$.

Exchange of vortices:

Vortex core carries an unpaired majorana mode. Transporting a fermion around a vortex yields -1 sign, therefore we expect that two MZMs associated with two different vortices transform as $\gamma_1 \to \gamma_2$, $\gamma_2 \to -\gamma_1$ under braiding (exchange). To see this, consider the guage invariant path operators associated with the two MZMs

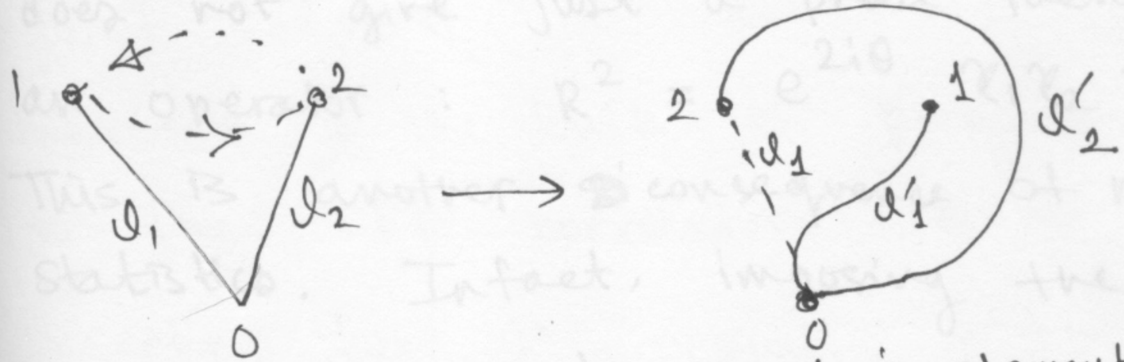


$$W(l_1) = c_0 \prod_0^1 u_{ij} \; \gamma_1$$

$$W(l_2) = c_0 \prod_0^2 u_{ij} \; \gamma_2$$

Where c_0 is the majorana fermion c at reference location 0 and γ_1, γ_2 are the two MZMs.

Under exchange, the path operators change as



Defining the operator that implements braiding as R,

$$R^+ W(l_1) R = W(l_1') = W(l_2)$$

[since the l's can be smoothly deformed]

$$R^+ W(l_2) R = W(l_2')$$

$$= W(l_1) \prod u_{ij}$$

closed path around vortex on the right

$$= - W(l_1)$$

since $\prod u_{ij} = -1$ as the path encloses a single vortex.

$$\Rightarrow \quad R^+ \gamma_1 R = \gamma_2$$

$$R^+ \gamma_2 R = - \gamma_1$$

$$\Rightarrow R = e^{i\theta} e^{-i\frac{\pi}{4} i\gamma_1\gamma_2} = e^{i\theta} \frac{[1+\gamma_1\gamma_2]}{\sqrt{2}}$$

where θ is still undetermined.

We see that doing the exchange twice does not give just a phase factor, but an operator : $R^2 = e^{2i\theta} \gamma_1\gamma_2 \neq \mathbb{1}$.

This is another consequence of non-abelian statistics. Infact, imposing the conditions, $R^\dagger \gamma_1 R \propto \gamma_2$ and $R^\dagger \gamma_2 R \propto \gamma_1$, there are just two unitaries which do the job, $R = e^{i\theta} \frac{[1+\gamma_1\gamma_2]}{\sqrt{2}}$ or $R = e^{i\varphi} \frac{[\gamma_1+\gamma_2]}{\sqrt{2}}$.

The second soln. is an abelian one since $R^2 \propto \mathbb{1}$. Ofcourse, in ~~the~~ our case, the first solution is realized.

Phase θ : Deriving θ is a bit of work. The answer is $\theta = \pi/8$. One quick way to see this is to use bulk-boundary correspondence between the topologically ordered bulk and the chiral CFT at the edge.

The anyon σ corresponds to the twist field σ in the CFT. The wave-fn in the bulk is proportional to the CFT correlators at the edge.

Inserting two σ operators in the bulk corresponds to $\langle \sigma(z)\, \sigma(0) \rangle$ correlation fn in the CFT $\langle \sigma(z)\, \sigma(0) \rangle \sim \dfrac{1}{z^{1/8}}$

Under exchange $z \to -z \Rightarrow$ one picks up a phase of $e^{i\pi/8}$ which is the abelian part of the statistics of σ (i.e. $\underset{=\pi/8}{\theta}$). Similarly $\langle \psi(z)\, \psi(0) \rangle \sim \dfrac{1}{z}$

$\Rightarrow$ ψ is a fermion since $z \to -z$ gives a phase of -1.

Consequences of non-abelian statistics

Consider four MZMs $\gamma_1, \gamma_2, \gamma_3, \gamma_4$. Let's initialize them so that γ_1, γ_2 fuse to a fermion and so do γ_3, γ_4.

Denoting $c_a = (\gamma_1 + i\gamma_2)/2$, $c_b = (\gamma_3 + i\gamma_4)/2$

this means that in the initial state $|0\rangle$,

$$c^{+}_a c_a |0\rangle = c^{+}_b c_b |0\rangle = |0\rangle$$

Denoting vacuum by $|\phi\rangle$, (i.e. $c_a|\phi\rangle = 0$, $c_b|\phi\rangle = 0$)

$$|0\rangle = c^{+}_a \; c^{+}_b \; |\phi\rangle$$

Let's act on $|0\rangle$ by R^{23}, the operator that exchanges γ_2 and γ_3.

$$R^{23}|0\rangle \propto \frac{1 + \gamma_2\gamma_3}{\sqrt{2}} |0\rangle$$

$$= \frac{|0\rangle}{\sqrt{2}} + \frac{\gamma_2 \gamma_3}{\sqrt{2}} |0\rangle$$

Now $\chi_2 \chi_3 |0\rangle$

$$= \chi_2 \chi_3 \frac{(\chi_1 - i\chi_2)}{2} \frac{(\chi_3 - i\chi_4)}{2} |\phi\rangle$$

$$= i|\phi\rangle$$

$$\Rightarrow R^{23}|0\rangle = \frac{|0\rangle + i|\phi\rangle}{\sqrt{2}}.$$

Thus, R^{23} rotates in the space of states spanned by $|0\rangle$ and $|\phi\rangle$.